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RATIONAL QUADRATIC X_1-SPLINE INTERPOLATION

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ABSTRACT

Parallel to the concept of X-splines, established by Clenshaw and Negus [1] for cubic or quadratic splines. We have developed in the present paper, the concept of rational X_r-splines. Further we have obtained a unique rational mid-point interpolatory X_1-spline of the type quadratic/quadratic. The convergence properties of the sequence of such rational splines have also been investigated.

KEYWORDS: *X*-Spline, Rational Spline, Convergence

1. INTRODUCTION

Cubic X-splines were introduced by Clenshaw and Negus [1] as a generalization of conventional cubic splines. The definition of cubic *X*-spline allowe the second derivative of the interpolant to possess discontinuities at the internal knots. Introducing one free parameter at each internal knot, the magnitudes of discontinuities of the second derivative are related in a simple manner to the magnitudes of discontinuities of the third derivative at the internal knots. It has been shown in [1] that the free parameters involved in the definition of *X*-splines may be chosen so as to impart desired geometric shape or analytic properties to the interpolating curve,

In the present paper, we have established a similar technique for the case of rational polynomial splines by defining rational X_r-splines, r = 1, 2…

A variety of rational splines have been studied by Gregory and Delbourgo [3], [4], and [5], Delbourgo [2], Ismail [6] and others (see [8] and [9] also). Rational splines have proved to be very useful and sometimes unavoidable tools for approximation of meromorphic functions or functions which are they rational functions. Further, it has been experienced (see [2] – [6[ ) that the rational splines are efficient shape preserving spline-interpolants for a monotonic and/or convex function.

Generally higher order smoothness is not achieved in case of rational lower order splines. The rational spline-interpolants studied earlier (c.f. [3], [4], or [5]) are in class C^1 [a, b] or in C^2 [a, b].

In view of this we have introduced in the present paper the concept of rational X_r-splines, r = 1, 2….which provide either case of computation or greater degree of freedom to control the shape of the interpolating curve. The rational X_r-splines in particular give the C^1 or C^2-splines studied earlier. We define X_r-splines in Section 2. In Section3, we develop a continuous X_1-rational spline with quadratic numerator and quadratic denominator. Convergence properties of class of this type of rational X_1-splines are studied in Section 4. Section 5 contains some remarks.

2. DEFINITIONS

Let P= $\{a = x_0 < x_1 < x_2 \ldots < x_n = b\}$ be a partition of a given interval [a, b].

We denote subinterval $[x_{i-1}, x_i]$ by I_i, for i=1,…, n-1and I_n stands for subinterval

$[x_{n-1},x_n]$. Let x_i-x_{i-1}=h_i, I =1 … n. Let s(.) be a piecewise rational polynomial of the type (p,q) defined over [a,b] so that its restriction s_i(.) in I_i is a rational function $P_i(x)/Qi(x)$, where Pi(x) is a polynomial of degree p and $Q_i(x)$ is a polynomial of degree q in I_i for each i.

Let $\alpha_i = \{\alpha_i\}_{i=1}^{n-1}$ be an (n-1)-tuple of real numbers. Piecewise rational polynomial of s (.) is rational X_r-splines with parameter vector α, if it satisfies following conditions:

s(.) $\in C^{(r-1)}$[a, b] so that

$s(x_i) = s_{i+1}(x_i)$, $s'(x_i) = s'_{i+1}(x_i)$, ………, $s_i^{(r-1)}(x_i) = s_{i+1}^{(r-1)}(x_i)$;

i = 1… n-1 (2.1)

And

$s_{i+1}^{(r)}(x_i) - s_i^{(r)}(x_i) = \alpha_i \{ s_{i+1}^{(r+1)}(x_i) - s_i^{(r+1)}(x_i) \}$.

i = 1,…, n-1 (2.2)

Therefore a piecewise rational function s(.) is a rational X_1-spline with parameter vector α if following conditions hold true:

S(.) $\in C^{0)}$[a, b] so that $s_i(x_i) – s_{i+1}(x_i)$ i = 1,…,n-1 (2.3)

And

$s'_{i+1}(x_i) – s'_i(x_i) = \alpha_i \{ s''_{i+1}(x_i) - s''_i(x_i) \}$. i = 1,…,n-1 (2.4)

The parameter α_i, i = 1, 2,…, n-1 introduced as above may be chosen appropriately to suit our demand of our interpolation problem. We note that if α_i = 0, i = 1, 2,…, n-1, we get rational splines in class C1[a, b]. Thus, rational X_1-splines are a natural generalization of conventional rational splines. In fact the significance of rational X_r-splines lies in the fact that we can impart desired twist in the shape of the interpolating curve by appropriate choice of the parameter vectorα. In the present paper we investigate the existence of mid-point-interpolatory rational X1-splines of the type (2, 2).

3. INTERPOLATION BY RATIONAL X1-SPLINE

Consider a piecewise rational polynomial s(.) such that in each sub interval Ii restriction si(.) of s is a rational function with a quadratic polynomial as its numerator and a quadratic polynomial as denominator. For xi-1$\leq$x$\leq$xi, we may consider the following convenient representation for s(x):

$$s(x)=\frac{(x-x_{i-1})s_i+(x_i-x)s_{i-1}+(x_i-x)(x-x_{i-1})c_i}{h_i+(x_i-x)(x-x_{i-1})}$$

$i=1,\ 2,\ldots,\ n$ (3.1)

Where si represents value of s (.) at xi and ci is arbitrary constant. We observe that

si (xi)=si+1 (xi) = si(say), i=1, 2, …,n-1

and hence s(x)$\in$ C0 [a, b]

Suppose that the piecewise rational polynomial s(.) satisfies the interpolatory condition :

$s(x_{i-1/2}) = f(x_{i-1/2}) = f_{i-1/2}$ (say) (3.2)

Where $x_{i-1/2} = (x_{i-1}+x_i)/2$.

Then we easily find that

$f_{i-1/2} = (2s_i+2s_{i-1}+h_ic_i)/(4+h_i)$

Thus ci= $[(4+h_i)f_{i-1/2}-2(s_i+s_{i-1})] / h_i$ (3.3)

Since s(.) is a rational X1-spline, it view of condition (2.4), we have

$[-s_{i+1}-s_i(3+h_{i+1})+(4+ h_{i+1})f_{i+1/2}]/h_{i+1}$

$-\ [s_i(h_i+3)+s_{i-1}-(4+h_i)f_{i-1/2}]/h_i$

$=2\alpha_i\ [\{s_{i+1}(2+ h_{i+1})+s_i(2+4h_{i+1}+ h^2_{i+1})$

$-\ (4+ h_{i+1})(1+ h_{i+1})\ f_{i+1/2}\}/\ h^2_{i+1}$

$-\{s_i(2+4h_i+h^2_i)+s_{i-1}(2+h_i)\ -\ (4+h_i)(1+h_i)\ f_{i-1/2}\}/\ h^2_i]$

Further a simple manipulation leads to the following:

$$\frac{s_{i+1}}{h_{i+1}}\left\{1+2\alpha_i\frac{(2+h_{i+1})}{h_{i+1}}\right\}$$

$$+s_i\left[\frac{(3+h_{i+1})}{h_{i+1}}+\frac{(3+h_i)}{h_i}+2\alpha_i\frac{(2+4h_{i+1}+h_{i+1}{}^2)}{h_{i+1}{}^2}-2\alpha_i\frac{(2+4h_i+h_i{}^2)}{h_i{}^2}\right]$$

$$+\frac{s_{i-1}}{h_i}\left\{1-2\alpha_i\frac{(2+h_i)}{h_i}\right\}$$

$$=(4+h_i)\frac{f_{i-1/2}}{h_i}\left\{1-2\alpha_i\frac{(1+h_i)}{h_i}\right\}+\frac{(4+h_{i+1})}{h_{i+1}}f_{i+1/2}\left\{1+2\alpha_i\frac{(1+h_{i+1})}{h_{i+1}}\right\} \qquad (3.4)$$

It is easy to see that the coefficient of si-1 is positive provided

$$\alpha_i < \frac{h_i}{2(2+h_i)} \tag{3.5}$$

Further we note that when (3.5) holds true for each i, the coefficient of s_i is also positive.

Now we consider the excess of the coefficient of si over the sum of those of si-1 and si+1. It is easy to conclude that this excess is

$$\frac{2+\mathrm{h}_{i+1}+2\alpha_{\mathrm{i}}(3+h_{i+1})}{h_{i+1}} + \frac{2+\mathrm{h}_{\mathrm{i}}-2\alpha_{\mathrm{i}}(3+h_i)}{h_i} = \delta_{\mathrm{i}} \text{ (say)} \tag{3.6}$$

Which is clearly positive provided

$$\frac{h_i}{2(2+h_i)} > \alpha_i > \frac{(h_i + h_{i+1} + h_i h_{i+1})}{3(h_{i+1} - h_i)} \quad \text{i= 1,2,…,n} \tag{3.7}$$

Therefore, the coefficient matrix of the system of equation (3.4) is diagonally dominant and hence is invertible, provided (3.7) holds true. Clearly the system of equations (3.4) then admits a unique solution. We have thus proved the following:

Theorem 3.1:

Let f be a 1-periodic function defined on

[a, b] and let P={a=x0<x1<x2<…<xn=b} be a partition of

[a, b]. Suppose $\alpha = \{\alpha_i\}^{n-1}{}_{i=1}$ is a parameter vector with entries as non-negative real numbers. Then there exists a unique rational (quadratic/quadratic) interpolatory X1-spline satisfying the interpolatory condition (2.4) provided (3.7) holds.

4. ERROR – ESTIMATES

In this Section we aim to obtain the error-estimates for the rational quadratic X1-splines established in Theorem 3,1. We suppose that the function f is a smooth enough. We denote by e(.) the error function s(.) – f (.).

Thus substituting ei+fi for si in (3.4) we get

$$\frac{e_{i+1}}{h_{i+1}}\left\{1+2\alpha_i\frac{(2+h_{i+1})}{h_{i+1}}\right\}$$

$$+e_i\left[\frac{(3+h_{i+1})}{h_{i+1}}+\frac{(3+h_i)}{h_i}+2\alpha_i\frac{(2+4h_{i+1}+h_{i+1}{}^2)}{h_{i+1}{}^2}-2\alpha_i\frac{(2+4h_i+h_i{}^2)}{h_i{}^2}\right]$$

$$+\frac{e_{i-1}}{h_{i-1}}\left\{1-2\alpha_i\frac{(2+h_i)}{h_i}\right\}$$

$$= (4+h_{i+1})\frac{f_{i+1/2}}{h_{i+1}}\left\{1+2\alpha_i\frac{(1+h_{i+1})}{h_{i+1}}\right\}+(4+h_i)\frac{f_{i-1/2}}{h_i}\left\{1-2\alpha_i\frac{(1+h_i)}{h_i}\right\}$$

$$-\frac{f_{i+1}}{h_{i+1}}\left\{1+2\alpha_i\frac{(2+h_{i+1})}{h_{i+1}}\right\}$$

$$-f_i\left[\frac{(3+h_{i+1})}{h_{i+1}}+\frac{(3+h_i)}{h_i}+2\alpha_i\frac{(2+4h_{i+1}+h_{i+1}{}^2)}{h_{i+1}{}^2}-2\alpha_i\frac{(2+4h_i+h_i{}^2)}{h_i{}^2}\right]$$

$$-\frac{f_{i-1}}{h_i}\left\{1-2\alpha_i\frac{(2+h_i)}{h_i}\right\}=\mathbf{R_i}\ (\text{say})$$

A simple manipulation shows that

$$\mathbf{R_i}=\frac{1}{h_{i+1}}\{f_{i+1/2}-f_{i+1}\}\ \frac{1}{h_i}\{\ f_{i-1/2}-f_{i-1}\}+\frac{(3+h_{i+1})}{h_{i+1}}\ \{f_{i+1/2}-f_i\}$$

$$+\frac{(3+h_i)}{h_i}\{\ f_{i-1/2}-f_i\}+2\alpha_i\frac{(2+h_{i+1})}{h_{i+1}{}^2}\{f_{i+1/2}-f_{i+1}\}$$

$$+2\alpha_i\frac{(2+4h_{i+1}+h_{i+1}{}^2)}{h_{i+1}{}^2}\{f_{i+1/2}-f_i\}+2\alpha_i\frac{(2+h_i)}{h_i^2}\{\ f_{i-1}-f_{i-1/2}\}$$

$$+\ 2\alpha_i\frac{(2+4h_i+h_i{}^2)}{h_i^2}\{\ f_i-f_{i-1/2}\}.$$

Therefore using the techniques of Sharma and Meir [7], we find that

$|ej| \leq \max|Rj/\delta\ j|$

$\delta_{\,i}|\mathbf{e_i}|\leq|\mathbf{R_i}|$

Where |ej| = max |ei| and δ i is given by (3.6)

$_{1}\leq{}_{i}\leq{}_{n}$

Therefore

$$\|e_i\|\leq\frac{(4+\bar{h})}{(2+\bar{h})}\left\{1+2\alpha_i\frac{(1+\bar{h})}{\underline{h}}\right\}$$

Where $\bar{h}$ = max hi, $\underline{h}$ = min hi.

$\mathbf{1 \leq i \leq n\ \ 1 \leq i \leq n}$

Hence we have proved the following:

Theorem 4.1:

If f is a 1–periodic function defined on

[0, 1] whose values at the mid points of mesh intervals of partition P are given and if s is the unique rational X1 interpolatory spline of Theorem 3.1 then

||ei|| $\leq$ K ω (f, h/2)

Where

$$K = \frac{(4+\bar{h})}{(2+\bar{h})}\left\{1+2\alpha_i \frac{(1+\bar{h})}{\underline{h}}\right\}$$

and ||.|| represents the row-max norm of vectors.

5 REMARKS

5.1: Theorem 3.1 establishes the unique existence of mid-point interpolatory rational quadratic X1-spline, while Theorem 4.1 provides the error-estimates for the rational quadratic spline of Theorem 3.1. Theorem 4.1 establishes that rational quadratic X1-spline of Theorem 3.1 is a good approximant to a 1-periodic function f$\in$ C0 [0, 1]. The rate of convergence of s to f is good and error approaches zero at a fast rate as $\bar{h} \to 0$.

5.2: If we choose parameters α_i = 0, i=1, 2, …, n-1, we get in Theorem 3.1, the rational splines studied in [3]. Hence rational X1–splines are a generalization of conventional rational splines.

5.3: We can get desired twist in the interpolating curve by suitable choice of parameter vector α.

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